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1989 J. Phys. A: Math. Gen. 22 1701

(http://iopscience.iop.org/0305-4470/22/10/023)

J. Phys. A: Math. Gen. 22 (1989) 1701-1703. Printed in the UK

COMMENT

On the simultaneous eigenproblem for the $x^2 - \lambda x^2(1+gx^2)^{-1}$ interaction: extension of Gallas' results

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Received 17 November 1988

Abstract. An ansatz for the eigenfunctions of the $x^2 - \lambda x^2(1+gx^2)^{-1}$ interaction has been recently introduced by Blecher and Leach. Using this ansatz, Gallas has obtained five solution pairs for $\lambda = -2g(3g+2)$. Gallas' results are generalised and extended here.

In a recent paper, Blecher and Leach (1987) discussed some solutions of the equation

$$\psi'' + [\varepsilon - x^2 - \lambda x^2(1+gx^2)^{-1}]\psi = 0 \tag{1}$$

for positive g and real x: this equation is of interest in several areas of physics. It was later shown by Gallas (1988) that the solution pairs reported by Blecher and Leach do not solve the simultaneous eigenproblem. Gallas went on to derive five solution pairs for which $\lambda = -2g(3g+2)$; this communication confirms and further extends his analysis.

Let us consider the twin solutions

$$\psi_a = x(1+gx^2)\exp(-x^2/2) \tag{2}$$

$$\psi_b = (1+gx^2)\exp(-x^2/2) \sum_{n\in\{0,N\}} \alpha_n x^{2n} \qquad N \geqslant 1,\ \alpha_0 = 1. \tag{3}$$

Substitution of (2) into (1), with $\varepsilon = \varepsilon_1$, easily leads to

$$\varepsilon_1 = 3 - 6g \tag{4a}$$

$$\lambda = -2g(3g+2) \tag{4b}$$

which correspond, respectively, to (3.2a) and (3.2c) of Gallas.

Next, substituting (3) into (1), with $\varepsilon = \varepsilon_2$, leads to the rather tedious equation

$$\begin{aligned}&\Bigg(\sum_{n\in\{0,N\}} \{\alpha_n x^{2n-2}[2n(2n-1)]\} + \sum_{n\in\{0,N\}} \{\alpha_n x^{2n}[g(2n+2)(2n+1) - (4n+1)]\}\\ &\quad + \sum_{n\in\{0,N\}} \{\alpha_n x^{2n+2}[1 - g(4n+5)]\} + \sum_{n\in\{0,N\}} \{\alpha_n x^{2n+4}[g]\}\Bigg)\\ &\quad + [\varepsilon_2 - x^2 - \lambda x^2(1+gx^2)^{-1}]\psi_b \exp(x^2/2) = 0.\end{aligned} \tag{5}$$

Equating the coefficients of like powers of x on the left-hand side of (5) to zero gives the three-term recursion relation

$$\begin{aligned}&2n(2n-1)\alpha_n - \alpha_{n-1}[(4n-3) - \varepsilon_2 - 2n(n-1)g]\\ &\quad - \alpha_{n-2}[(4n-3)g - g\varepsilon_2 + \lambda] = 0 \qquad n = 1, 2, 3, \ldots, N+2\end{aligned} \tag{6}$$

0305-4470/89/101701+03\$02.50

with $\alpha_n = 0$ for $n < 0$, and $\alpha_0 = 1$ for normalisation. After noting that $\alpha_{N+1} = \alpha_{N+2} = 0$, and substituting $n = N+2$ in (6), one obtains

$$(4N+5)g - g\varepsilon_2 + \lambda = 0. \tag{7a}$$

From (4*b*) and (7*a*) one also gets

$$\varepsilon_2 = 4N + 1 - 6g. \tag{7b}$$

Similarly, the use of $n = 1$ in (6) results in

$$\varepsilon_2 = 1 - 2g - 2_1. \tag{8a}$$

Elimination of ε_2 between (7*b*) and (8*a*) results in

$$\alpha_1 = 2(g - N). \tag{8b}$$

Finally, using the recursion relation (6), $\alpha_0 = 1$, and (8*b*), it becomes possible to obtain α_n for $3 \leqslant n \leqslant N$.

Now let us come to the polynomial equations for g for given N. The relation

$$4g(m+2)\alpha_{N-m-2} + \alpha_{N-m-1}[4(m+1) + 2g(N-m+1)(2N-2m-3)] + \alpha_{N-m}[2(N-m)(2N-2m-1)] = 0 \tag{9}$$

can be obtained by using $n = N - m$ in (6). From (9), it can be shown that the ratio

$$F(N-m;\, N-m-1) = -4(m+1)g[4m + 2g(N+2-m)(2N-2m-1) + 2(N-m+1) \times (2N-2m+1)F(N-m+1;\, N-m)]^{-1} \qquad N-2 \geqslant m \geqslant 1 \tag{10a}$$

where the definition

$$F(i, j) = \alpha_i / \alpha_j \tag{10b}$$

has been used. It should also be noted that

$$F(i;\, i-1) = F(N - [N-i];\, N - [N-i] - 1). \tag{10c}$$

Repeated application of (10*a*) and (10*c*) to the ratio α_1/α_0 leads to the continued fraction

$$\alpha_1/\alpha_0 = \frac{1}{2}\left(\frac{2\times 1\times 1[-4(N)g]}{[4(N-1) + 2\times 3\times 1g] +}\,\frac{2\times 2\times 3[-4(N-1)g]}{[4(N-2) + 2\times 4\times 3g] +}\,\frac{2\times 3\times 5[-4(N-2)g]}{[4(N-3) + 2\times 5\times 5g] +}\,\frac{2\times 4\times 7[-4(N-3)g]}{[4(N-4) + 2\times 6\times 7g] + \ldots}\,\ldots\frac{2N(2N-1)[-4g]}{[4(N-N) + 2(N+2)(2N-1)g]}\right) \tag{11}$$

which is of finite size. But, using $\alpha_0 = 1$ and (8*b*), the ratio α_1/α_0 can also be obtained as

$$\alpha_1/\alpha_0 = 2(g - N). \tag{12}$$

Equating the right-hand sides of (11) and (12) gives a polynomial equation of order N in g, which has to be solved in order to get the $(2N)$th excited state.

In conclusion, it should be noted that the five solution pairs given by Gallas can be obtained from the aforementioned analysis by using $N = 1, 2, 3, 4$ and 5, and the present work constitutes an extension of previous analysis (Gallas 1988).

References

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